

Equilibrium of anchored interfaces with quenched disordered growth

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The roughening behavior of a one-dimensional interface fluctuating under quenched disorder growth is examined while keeping an anchored boundary. The latter introduces detailed balance conditions which allows for a simple but thorough analysis of equilibrium aspects at both macroscopic and microscopic scales. It is found that the interface roughens linearly with the substrate size only in the vicinity of special disorder realizations. Otherwise, it remains stiff and tilted.

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Studies of inhomogeneous interface growth have thrived in a variety of physical contexts [1] characterizing phenomena as diverse as fluid imbibition in porous media [2] and crystalline surface growth on disordered substrates [3]. It is known that in these situations a small amount of disorder can severely modify the interface motion and ultimately alter its roughening behavior [1]. This is the case of time independent but spatially random growth rates arising, for instance, from a quenched array of columnar defects pinning the flux lines, here playing the role of interfaces, in dirty high temperature superconductors [4]. Another mechanism whereby the interface character results deeply affected at large times is realized by anchoring conditions which suppress fluctuations at the interface boundaries. Such confined geometries actually occur in the unbinding of polymers from a wall [5], and also emerge as domain walls of (2+1)-dimensional cellular automaton models [6], as well as in stationary nonequilibrium systems in $d=1$ [7]. In this work we focus on the combined effect that quenched disorder and anchoring conditions brings about in the steady state (SS) properties of one-dimensional (1D) interfaces, at both macroscopic and microscopic scales. This extends the SS results obtained in a related study [8] where the interplay between a single growth inhomogeneity and anchored boundaries was considered.

The problem is most conveniently treated in the discrete formulation of growth processes. As usual [9], here we represent the latter in terms of restricted solid on solid (RSOS) configurations of heights $(h_0), h_1, \dots, h_L$ growing stochastically on a substrate of size L . Throughout the evolution, fluctuations are suppressed entirely at h_0 , whereas all other h_j can increase (decrease) in two height units with substrate dependent rates ϵ_j (ϵ'_j) [10]. Due to the RSOS constraints $|h_{j+1} - h_j| = 1$, these variations can only occur at local extrema of the interface, as illustrated in Fig. 1. Despite its simplicity, it will turn out that this model encompasses a disorder driven transition along with unusual roughening exponents alike those produced by isolated impurities [8].

Following most studies [1], we concentrate on the root mean square dispersion of heights, commonly associated to the width W of the system. Under quenched growth this further requires the averaging of W over all disorder realizations (denoted by the brackets below), namely,

$$\langle W(L, t) \rangle = \left\langle \left\{ \frac{1}{L} \sum_j \overline{[h_j(t) - H(t)]^2} \right\}^{1/2} \right\rangle, \quad (1)$$

where $H(t)$ is the mean interface height, and the overbar indicates an ensemble average over all possible evolution histories up to time t . On general grounds [1,11], it can be argued that $\langle W(L, t) \rangle$ scales as $L^\zeta f(t/L^\zeta)$ with a universal scaling function behaving as $f(x) \sim x^{s/z}$ for $x \ll 1$, whereas for $x \gg 1$ it remains constant. Consequently, for $t \gg L^\zeta$ the width saturates as L^ζ while growing as $t^{s/z}$ at much earlier stages. In what follows we content ourselves with studying just the former situation, i.e., the stationary regime controlled by the roughening exponent s . In contrast to the dynamic exponent z , s lends itself more readily for a thorough analysis under growth disorder, so hereafter we shall work over Eq. (1) directly in the limit $t \rightarrow \infty$.

a. *Particle representation.* To this aim, let us first consider the conditions that the SS imposes on the above processes. For ease of analysis we retain the interface slopes $s_j = h_j - h_{j-1} = \pm 1$ rather than its heights, or alternatively the set of occupation numbers $\{n\} \equiv \{(1+s)/2\}$ conforming a height configuration. This enables to exploit the well known mapping [1,9] from a 1D RSOS interface to a driven particle system, i.e., $h_j = \sum_{i \leq j} (2n_i - 1)$, here specially adapted to account for both anchored and free edges as well as for quenched disorder. In particular, height variations at the free boundary translate here into injection and ejection of particles, as is easily visualized in Fig. 1. In this latter representation one can readily build up a SS probability measure $P(\{n\})$ satisfying detailed balance, regardless the specific realization of disorder considered. In equilibrium, the bulk and right boundary processes will demand, respectively (see Fig. 1)

$$\epsilon'_i P(n_1, \dots, \underbrace{1, 0}_{i, i+1}, \dots, n_L) = \epsilon_i P(n_1, \dots, \underbrace{0, 1}_{i, i+1}, \dots, n_L), \quad (2)$$

$$\epsilon'_L P(n_1, \dots, n_{L-1}, 1) = \epsilon_L P(n_1, \dots, n_{L-1}, 0), \quad (3)$$

and therefore the SS distribution must be of the form $e^{-\sum_j V_j n_j}$. Specifically, Eq. (2) entails a hard-core particle potential $V_j = V_1 + \sum_{i < j} \ln(\epsilon_i / \epsilon'_i)$, whereas Eq. (3) fixes the

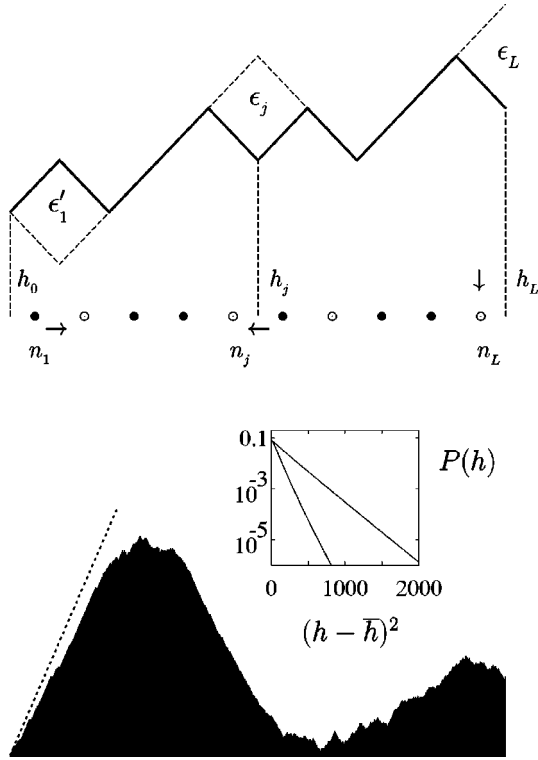


FIG. 1. RSOS interface anchored at h_0 , thought of as an asymmetric exclusion process in which height slopes $s_j = h_j - h_{j-1}$ are associated to particle occupations $n_j = \frac{1}{2}(1 + s_j)$. At the free edge h_L variates in 2 (-2) with rate ϵ_L (ϵ'_L) by particle deposition (evaporation), whereas for $1 \leq j < L$ height changes correspond to left (right) particle hoppings with rates ϵ_j (ϵ'_j). Lower panel: typical snapshot for $L = 10^3$ after 10^5 steps per height using a binary disorder with $\epsilon' / \epsilon = 0.8, 1.2$ under condition (8), i.e., for $p = p_c$ in the text. Otherwise, configurations become tilted (dotted line). Inset: Gaussian distribution of heights for $j = 100$, at and slightly below p_c (upper and lower lines).

value of the additive constant. Equivalently, by introducing the total particle number $\mathcal{N} = \sum_j n_j$, we thus obtain a sample dependent grand canonical form $P(\{n\}) = (\mu^{\mathcal{N}} / Z) e^{-\sum_j V_j n_j} (V_1 \equiv 0)$, with a particle fugacity $\mu = \prod_j \epsilon_j / \epsilon'_j$ and a normalizing partition function $Z = \prod_j (1 + \mu e^{-V_j})$. A similar SS form appeared in Ref. [8], although constrained throughout by a *fixed* number of particles as imposed by the *two* anchored edges considered there. By restricting to a single anchored boundary, here we facilitate significantly the analysis of all SS aspects discussed subsequently, though at the expense of the nonequilibrium approach given in Ref. [8]. For posterior use, it can be easily checked that the particle densities \bar{n}_i involved in the calculation of height profiles are

$$\bar{n}_i = \left(1 + \prod_{j \geq i} \epsilon'_j / \epsilon_j \right)^{-1}, \quad (4)$$

while the pair correlations needed for the analysis of height-height correlators appearing in W , result totally decoupled, that is, $\bar{n}_i \bar{n}_j = \bar{n}_i \bar{n}_j \forall i \neq j$ and for all disorder realizations. However as we shall see below, there is a particular situa-

tion, namely, $\mu \rightarrow 1$, for which height variables become strongly correlated.

b. *Height distribution.* Before embarking in the evaluation of the double average of Eq. (1), we should determine, first, whether the growth of a given sample can actually produce a rough interface. It may well happen that while the width increases with the typical substrate size, the whole system becomes actually smooth, as is the case of the linear profile exhibited in Fig. 1. In analyzing this issue we focus attention on a more microscopic level of description such as the *single* height probability density $P(h_N)$ at a given location N . In turn, this requires the evaluation of an M -particle partition function $Z_M \equiv \omega_M / M!$ constructed as

$$\omega_M = \mu^M \sum_{j_1 \cdots j_M \leq N} ' e^{-V_{j_1}} \cdots e^{-V_{j_M}}, \quad (5)$$

where Σ' restricts the sums to $\{j_1 \neq \cdots \neq j_M\} \in [1, N]$ so as to keep a constant number of particles within that interval. Clearly, this M -body quantity ($M \leq N$) is associated to the wanted probability at $h_N = 2M - N$, since by construction $P(2M - N) = Z_M / [\prod_{j=1}^N (1 + \mu e^{-V_j})]$. To account for the hardcore interactions we build up a recursive relation in the particle number M . Introducing the auxiliary functions $g(k) = \mu^k \sum_{j=1}^N e^{-kV_j}$, this can be achieved via the identity $\omega_M = g(1) \omega_{M-1} - (M-1) S_{M-2}(2)$, where the latter factor is defined generically as $S_{M-k}(k) = \mu^M \sum_{j_1 \neq \cdots \neq j_{M-k}} e^{-V_{j_1}} \cdots e^{-V_{j_{M-k}}}$. In particular, $S_{M-k}(2)$ involves constrained sums having $\{j \neq j_1 \neq \cdots \neq j_{M-2}\} \in [1, N]$ and subtracts the unwanted terms resulting from $(M-1)$ double occupation configurations included in $g(1) \omega_{M-1}$. Using a similar criteria, we may also infer that $S_{M-2}(2) = g(2) \omega_{M-2} - (M-2) S_{M-3}(3)$, where the last term now cancels triple occupation contributions to $S_{M-2}(2)$. Thereafter, iterating this reasoning down to $M=1$, it can be straightforwardly verified that in terms of the j -particle partition functions $Z_j = \omega_j / j!$ we are left with

$$Z_M = \frac{(-1)^{M+1}}{M} \left[g(M) + \sum_{j=1}^{M-1} (-1)^j g(M-j) Z_j \right], \quad (6)$$

where $Z_1 = g(1)$. A simple numerical evaluation of such recurrence thus enables us to examine the height distribution, particularly at the tails where statistics become more demanding. Despite the noisy particle potential V of individual samples it turns out that for $M \gg 1$ the distribution itself collapses towards a universal—Gaussian—form. This is evidenced by the semilog inset of Fig. 1, for instance, in the case of a binary concentration p of growth rates with probability $p \delta[(\epsilon' / \epsilon) - a] + (1-p) \delta[(\epsilon' / \epsilon) - b]$. Also, recursion (6) reveals that in most cases the interface is characterized by a *tilted* profile $\bar{h}_N \sim \pm N$ whose local height fluctuations (HF) set out to be quite *small*, even far away from the anchored boundary h_0 , i.e., the interface is *stiff*, in agreement with the linear snapshot displayed in Fig. 1. On approaching certain disorder conditions however, Gaussian tails widen significantly (slope diminishing shown in the inset), so HF take over and the stiff regime no longer holds. This is illustrated by the rough profile also exhibited in Fig. 1.

c. *Roughening criterion.* It is therefore important to focus on these special conditions so as to ensure a meaningful result in the calculation of Eq. (1). Let us then consider the fluctuations of the total number of particles $\langle \sigma^2 \rangle \equiv \langle \mathcal{N}^2 - \bar{\mathcal{N}}^2 \rangle$, averaged over a generic growth disorder. For equilibrium regimes, this is a representative quantity to look at, as in terms of the occupation densities it is immediate to check that it coincides with the height variance at the free edge of the interface, i.e., $4\langle \sigma^2 \rangle = \langle \bar{h}_L^2 - \bar{h}_L^2 \rangle$ whereas by construction it constitutes an upper bound for all HF. To facilitate its analysis and by virtue of the products involved in Eq. (4), it is convenient to introduce the rate disorder variables $u_i = \ln(\epsilon'_i / \epsilon_i)$, in terms of which these fluctuations are casted as $\langle \sigma^2 \rangle = \frac{1}{4} \sum_{j=1}^L \langle [\cosh(\frac{1}{2} \sum_{i=j}^L u_i)]^{-2} \rangle$. Assuming a site independent distribution of growth ratios, say with mean $\langle u \rangle$ and variance s^2 , we can thus exploit the central limit theorem [12] for these new variables and carry out the disorder average. Hence, it can be readily shown that

$$\langle \sigma^2 \rangle \simeq \sum_{j=1}^k \alpha_j + \frac{1}{4s} \sum_{j=k+1}^L \int_{-\infty}^{+\infty} \frac{\exp\left[-\frac{(u-j\langle u \rangle)^2}{2js^2}\right]}{\sqrt{2\pi j} \cosh^2\left(\frac{u}{2}\right)} du, \quad (7)$$

where $\alpha_j = \langle \bar{n}_{L+1-j}(1 - \bar{n}_{L+1-j}) \rangle$, and k is eventually a large but finite integer ($k \ll L$). The point to emphasize here is that whatever disorder distribution is considered, the interface fluctuations are intrinsically dominated by the above series of integrals. Thus, apart from a bounded quantity $0 \leq \sum_{j=1}^k \alpha_j \leq k/4$, the effect of any distribution on $\langle \sigma^2 \rangle$ will parallel the one caused by the Gaussian disorder. On the other hand, as long as $s \neq 0$ the asymptotic analysis of Eq. (7) shows that the value of $\langle u \rangle$ is crucial, since for large L

$$\langle \sigma^2 \rangle \propto \begin{cases} \sqrt{L}, & \text{if } \langle \ln \frac{\epsilon'}{\epsilon} \rangle = 0 \\ \text{bounded,} & \text{otherwise,} \end{cases} \quad (8)$$

which finally provides the disorder conditions able to produce a rough interface. In common with this result, HF around a single growth inhomogeneity as that considered in Ref. [8] can also grow unbounded insofar $\epsilon = \epsilon'$ throughout the bulk. However, for that situation HF turn out to increase linearly with the substrate size [8]. As was mentioned earlier, roughening occurs for fugacities such that $\mu \rightarrow 1^\pm$ in turn favoring strong fluctuations in the particle occupation numbers. In that case, $|\bar{s}_i| < 1$ and height variables become tightly correlated along the substrate since $\bar{h}_i \bar{h}_j - \bar{h}_i \bar{h}_j = (i - \sum_{k=1}^i \bar{s}_k^2) \neq 0, \quad \forall i < j$. Otherwise, the slopes $\bar{s}_i \rightarrow -1, 1$ (i.e., $\bar{N}/L \rightarrow 0, 1$), so the interface becomes tilted and hardly flexible because $\langle \bar{h}_i^2 - \bar{h}_i^2 \rangle \leq 4\langle \sigma^2 \rangle \forall i$ is bounded. To illustrate the effectiveness of the above criterion for non-Gaussian cases, we address to the binary distribution referred to above. Using the binomial distribution to weight all possible forms in which the growth ratios a, b may show up on a segment of length l , we find

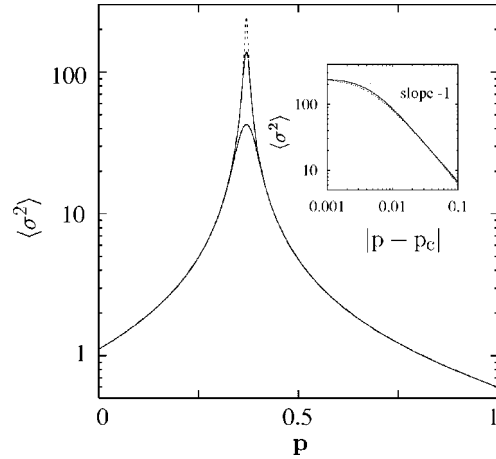


FIG. 2. Particle fluctuations arising from a binary growth distribution. Data for $a=0.5, b=1.5$ while varying the concentration of a . Curves from top to bottom denote in turn results for $L=10^5, 10^4$ and 10^3 . The inset exhibits the same algebraic divergence (dashed slope) either nearing $p_c = [1 - \ln a / \ln b]^{-1}$ from below or above (dotted and solid lines closely following each other; $L=10^5$).

$$\langle \sigma^2 \rangle = \sum_{l=1}^L \sum_{k=0}^l \binom{l}{k} \frac{p^k (1-p)^{l-k}}{4 \cosh^2 \varphi_{l,k}}, \quad (9)$$

where $2\varphi_{l,k} \equiv k \ln a + (l-k) \ln b$. Figure 2 shows the typical behavior of this quantity for $a < 1 < b$. In agreement with condition (8), there exists a critical concentration $p_c = [1 - \ln a / \ln b]^{-1}$, i.e., $\langle u \rangle = 0$, for which particle fluctuations increase as \sqrt{L} (see inset of Fig. 3), while in the thermodynamic limit diverge as $|p - p_c|^{-1}$ as indicated by the inset of Fig. 2. In nearing p_c from below (above), this disorder-driven transition brings the interface from a tilted state with slope -1 ($+1$) to a rough phase in a continuous manner. Clearly, for $b < 1 < a$ the rapprochement direction is

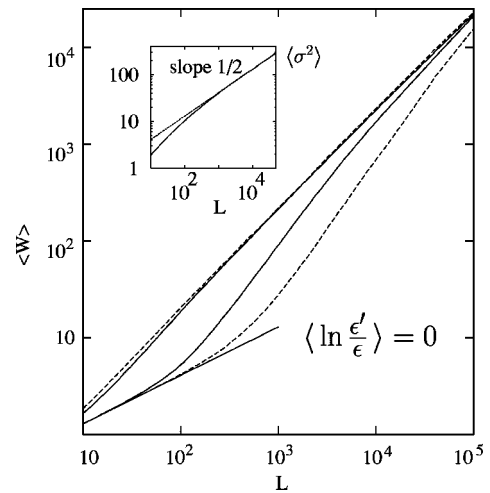


FIG. 3. Quenched interface width scaling linearly with the substrate size. Averages were taken over Gaussian and binary distributions (dashed and solid lines, respectively), using condition (8). Curves in descending order stand in turn for $s=1, p_c=0.4, s=0.01$, and $p_c=0.004$ ($a=0.5$); the lowest slope is $1/2$. The inset displays fluctuations (9) scaling as $L^{1/2}$ (dashed slope) for $p_c=0.4$.

inverted, otherwise the system remains tilted. It is of interest to note here that criterion (8) implies that locally, uniform interfaces ($s=0$) can fluctuate stronger than disordered ones. In fact, for $\epsilon'_i = \epsilon_i$ all interface configurations are equally likely so the single height probability density derived from Eq. (5) reduces to $P(h) \sim 1/\sqrt{L} \exp(-h^2/2L)$. So, in the pure substrate $\sigma^2 \propto L$, rather than diverging as in Eq. (8). What remains to be seen is whether different scaling behaviors can also emerge on a more macroscopic description, such as the width of Eq. (1) and towards which we finally turn.

d. *Width behavior.* Let us then consider this global quantity in the slope representation discussed throughout. After some algebraic steps, it can be readily demonstrated that the width of a given sample is expressed as

$$W^2(L) = \frac{(L^2 - 1)}{6L} + \frac{2}{L^2} \sum_{i < j} (L + 1 - j)(i - 1) \overline{s_i s_j}. \quad (10)$$

In particular, for equilibrium regimes there is a key simplification in the average of W , as $\overline{s_i s_j} = \overline{s_i} \overline{s_j}$ over all disorder realizations.

For binary rates, a closed though rather involved expression of $\langle \overline{s_i s_j} \rangle$ can be easily found upon recurring once more to the binomial distribution. By inserting that expression in Eq. (10), the size dependence of the quenched width is evaluated numerically and plotted in Fig. 3 (solid lines), using the critical concentrations p_c required by criterion (8). Clearly, the results support a power law growth of $\langle W \rangle$ consistent with a rather unusual roughening exponent $\zeta \approx 1$ extended over more than three decades. Thus, we see that in sharp contrast to what occurs at the microscopic scale where HF of disordered interfaces are weaker ($\langle \sigma^2 \rangle \propto \sqrt{L}$) than uniform ones ($\sigma^2 \propto L$), macroscopically the former can grow rougher than the latter. Though as pointed out above HF around isolated inhomogeneities scale more rapidly, we also

see that for large length scales finite disorder concentrations amount to a single impurity as both situations are typified by the same ζ value [8], a feature which is not evident *a priori*.

On approaching weakly disordered regimes however, size effects become quite pronounced. Even for large substrate lengths $\langle W \rangle$ grows with the $\zeta=1/2$ exponent characteristic of homogeneous systems [9,13] though asymptotically the linear scaling is recovered. Hence, at the static level of the ζ -exponents there is a *discontinuous* scaling regime at $p_c = 0^+, 1^-$ where $\epsilon \approx \epsilon'$ for most sites. In fact, the numerical estimation of $\langle W \rangle$ via Eqs. (4) and (10) over a Gaussian growth distribution yields the same value of ζ while also implying a discontinuous behavior near the homogeneous case, when $s \rightarrow 0^+$. This is indicated by the dashed lines of Fig. 3 after averaging over 10^4 samples. As was discussed in Ref. [8], such discontinuous feature also emerges under a single bulk inhomogeneity of even arbitrarily small but non-vanishing strength.

To summarize, we have constructed a recursive relation [Eq. (6)] which allowed us to examine disordered interfaces at microscopic scales. For most situations this yielded a stiff picture given the small HF obtained, though in the vicinity of special disorder realizations HF take over and wipe out the interface stiffness. This led us to propose a roughening criterion by looking at the fluctuations of the total particle number along with their divergence conditions [Eq. (8)]. Within such regimes we analyzed the scaling behavior of quenched widths using both Gaussian and non-Gaussian disorders. The corresponding results support a linear growth of $\langle W \rangle$ with the substrate size, i.e., $\zeta \approx 1$ (Fig. 3), which possibly reflects a tendency of inhomogeneous interfaces to crumple on large scales [1,8].

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